Modeling Correlation in the Face of Uncertain Parameters

Stephen J. Mildenhall
Aon Singapore Innovation Center

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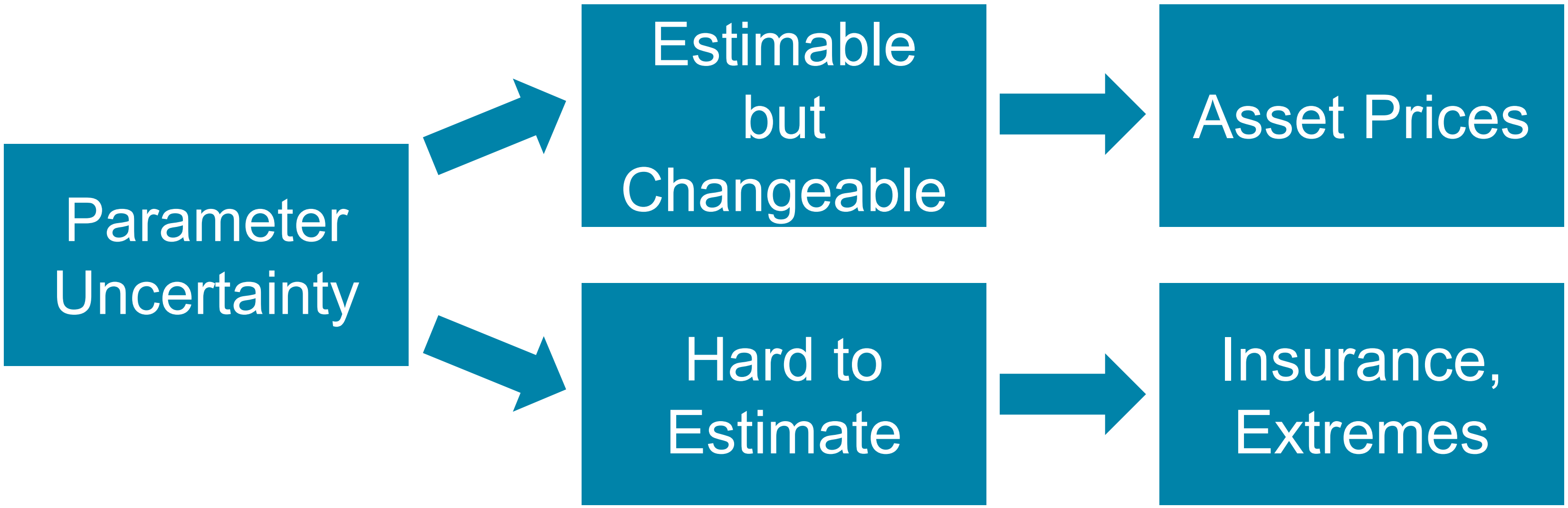
Motivation



- Manifestations of parameter uncertainty
  - Assets: option pricing, realized vs. implied volatility, index options
  - Insurance: catastrophe reinsurance, clash, desire for aggregate stop loss covers
- Quantity of parameters vs. quantity of data
- Interplay of univariate volatility and correlation

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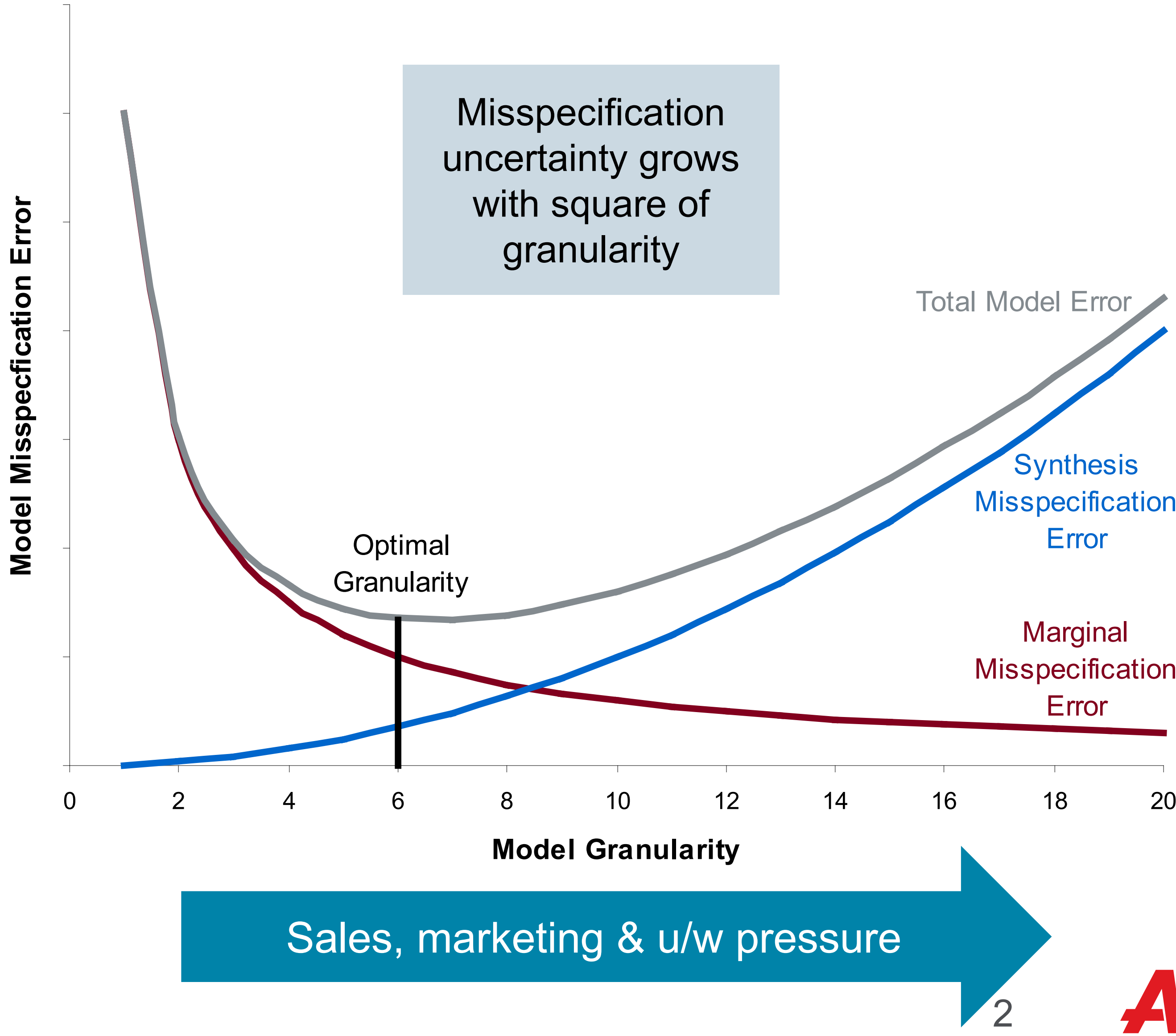
Correlation Eventually Dominates Uncertainty



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The Importance of Uncertainty

- Fundamental purpose of capital modeling is assessing uncertainty
- Uncertainty is priced
- Insidious uncertainty is priced more
  - Uncertainty that is statistically difficult to differentiate from the model assumptions
  - Catastrophe model uncertainty is a classic example
  - Long-term return on stocks is another example
- Underwriters and investors often act making “worst case” assumptions, worst case being that most worrying most worrying to them
  - Event frequency
  - Model miss, demand surge, unmodeled perils, unknown exposures
  - Systemic risks, supply chain, mass tort, law change
  - Contagion, correlation in portfolios, clash

Evidence

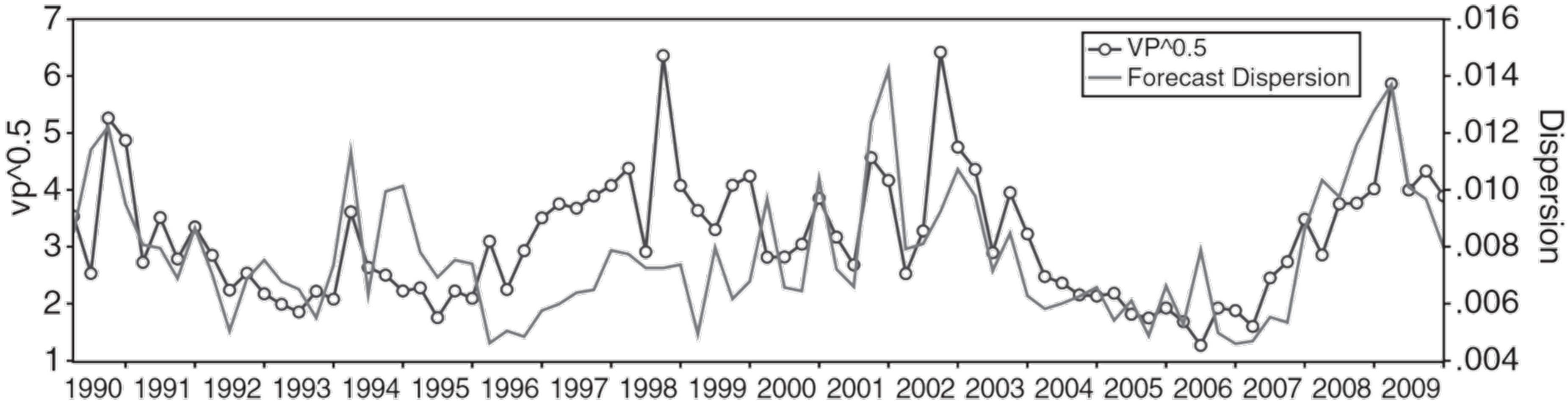

Figure 1. Forecast dispersion and variance premium. The figure plots the standard deviation in forecasts of next quarter's real GDP growth from the Survey of Professional Forecasters (SPF) versus the square root of VP at the end of the previous quarter. The sample is 1990Q1 to 2009Q4. The correlation of the two series is 0.54 with a robust standard error of 0.14.

- VP is the variance premium, the difference between option implied volatility and realized volatility over one month, or E^Q(Vol)-E^P(Vol)
- Asset volatility compounds individual stock volatility and correlation between stocks
- 1998 miss is the Asian debt crisis

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Source for graphic and concept: Drechsler, I., Uncertainty, Time-Varying Fear, and Asset Prices, J of Fin, Oct 2013

How to Measure Correlation, Dependency

Pearson Correlation Cov(X,Y)/SD(X)SD(Y)	**Rank Correlation** Correlation of Ranks = “Copula Correlation”
Correlation of Normal Transformed Data Normal Transform: $Phi^{-1}(F(X))$ $Phi^{-1}(Rank/(n+1))$	**Kendall’s Tau** 2 (#Concordant Pairs – #Discordant Pairs) / n(n-1)

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How to Measure Correlation: Quiet Period

Pearson Correlation

	AA	DOW	XOM	C	CB	FNMA	K	KO	PG	AET	GE
AA		0.52	0.30	0.40	0.27	0.25	0.22	0.09	0.20	0.15	0.39
DOW			0.35	0.40	0.37	0.21	0.36	0.12	0.21	0.11	0.45
XOM				0.49	0.38	0.31	0.34	0.35	0.35	0.19	0.46
C					0.34	0.44	0.33	0.28	0.41	0.21	0.45
CB						0.21	0.26	0.15	0.21	0.05	0.43
FNMA							0.17	0.13	0.31	0.07	0.28
K								0.20	0.31	0.08	0.26
KO									0.24	0.17	0.23
PG										-0.00	0.37
AET											0.16
GE											

Rank Correlation

	AA	DOW	XOM	C	CB	FNMA	K	KO	PG	AET	GE
AA		0.52	0.33	0.42	0.28	0.28	0.19	0.12	0.26	0.21	0.41
DOW			0.38	0.41	0.40	0.26	0.35	0.18	0.28	0.14	0.47
XOM				0.52	0.37	0.32	0.33	0.33	0.41	0.20	0.49
C					0.37	0.43	0.30	0.30	0.47	0.23	0.46
CB						0.29	0.29	0.19	0.27	0.23	0.44
FNMA							0.19	0.16	0.33	0.11	0.28
K								0.25	0.37	0.12	0.25
KO									0.26	0.16	0.30
PG										0.07	0.43
AET											0.17
GE											

Correlation of Normal Transform

	AA	DOW	XOM	C	CB	FNMA	K	KO	PG	AET	GE
AA		0.53	0.33	0.39	0.27	0.28	0.20	0.11	0.24	0.19	0.40
DOW			0.39	0.42	0.39	0.23	0.38	0.16	0.24	0.13	0.47
XOM				0.51	0.37	0.32	0.36	0.36	0.37	0.19	0.48
C					0.35	0.45	0.33	0.30	0.42	0.24	0.45
CB						0.24	0.29	0.19	0.23	0.15	0.44
FNMA							0.18	0.15	0.33	0.10	0.28
K								0.24	0.36	0.10	0.29
KO									0.26	0.15	0.28
PG										0.03	0.38
AET											0.18
GE											

Kendall's Tau

	AA	DOW	XOM	C	CB	FNMA	K	KO	PG	AET	GE
AA		0.37	0.23	0.28	0.20	0.19	0.13	0.08	0.17	0.15	0.28
DOW			0.27	0.28	0.27	0.17	0.24	0.13	0.18	0.09	0.32
XOM				0.36	0.25	0.22	0.23	0.23	0.27	0.14	0.34
C					0.25	0.30	0.21	0.20	0.33	0.16	0.32
CB						0.20	0.20	0.13	0.18	0.16	0.31
FNMA							0.13	0.11	0.22	0.07	0.19
K								0.17	0.25	0.08	0.17
KO									0.18	0.11	0.21
PG										0.04	0.30
AET											0.11
GE											

- Correlation is estimated from the period, June 23, 2003 to April 7, 2004

How to Measure Correlation: Financial Crisis

Pearson Correlation

	AA	DOW	XOM	C	CB	FNMA	K	KO	PG	AET	GE
AA		0.58	0.67	0.43	0.49	0.22	0.54	0.50	0.52	0.43	0.58
DOW			0.57	0.45	0.51	0.14	0.52	0.39	0.55	0.42	0.58
XOM				0.39	0.59	0.13	0.58	0.55	0.68	0.45	0.51
C					0.53	0.31	0.34	0.31	0.39	0.45	0.59
CB						0.16	0.52	0.47	0.56	0.50	0.58
FNMA							0.11	0.08	0.09	0.15	0.21
K								0.64	0.68	0.47	0.47
KO									0.61	0.43	0.38
PG										0.49	0.51
AET											0.45
GE											

Rank Correlation

	AA	DOW	XOM	C	CB	FNMA	K	KO	PG	AET	GE
AA		0.58	0.60	0.40	0.45	0.33	0.37	0.37	0.38	0.31	0.58
DOW			0.57	0.49	0.54	0.33	0.45	0.42	0.52	0.38	0.62
XOM				0.38	0.54	0.28	0.46	0.45	0.57	0.38	0.53
C					0.55	0.55	0.34	0.33	0.37	0.40	0.58
CB						0.40	0.52	0.46	0.53	0.47	0.58
FNMA							0.22	0.21	0.22	0.27	0.41
K								0.55	0.62	0.39	0.43
KO									0.55	0.34	0.41
PG										0.39	0.50
AET											0.42
GE											

Correlation of Normal Transform

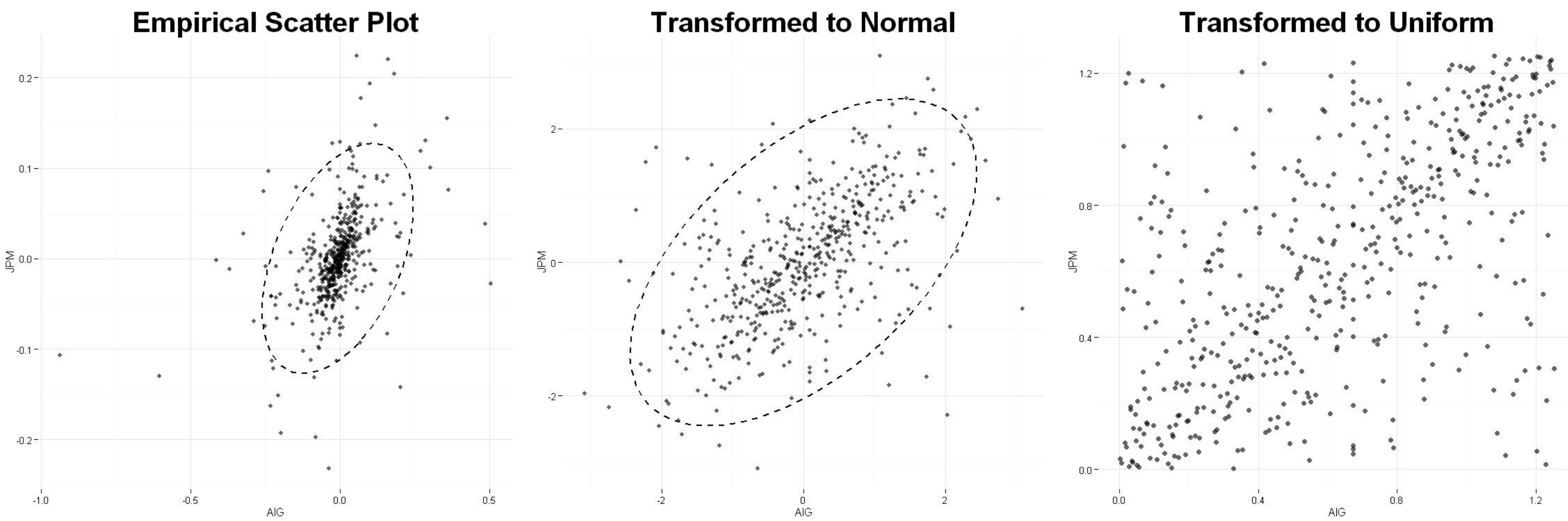
	AA	DOW	XOM	C	CB	FNMA	K	KO	PG	AET	GE
AA		0.59	0.63	0.44	0.48	0.32	0.45	0.43	0.45	0.37	0.59
DOW			0.59	0.49	0.55	0.30	0.49	0.42	0.55	0.41	0.62
XOM				0.42	0.58	0.27	0.51	0.48	0.61	0.41	0.55
C					0.56	0.54	0.36	0.35	0.41	0.44	0.60
CB						0.36	0.53	0.47	0.56	0.50	0.60
FNMA							0.23	0.20	0.21	0.28	0.40
K								0.59	0.65	0.43	0.46
KO									0.58	0.39	0.42
PG										0.44	0.52
AET											0.44
GE											

Kendall's Tau

	AA	DOW	XOM	C	CB	FNMA	K	KO	PG	AET	GE
AA		0.42	0.43	0.29	0.31	0.22	0.26	0.26	0.27	0.22	0.43
DOW			0.41	0.35	0.39	0.22	0.32	0.29	0.37	0.27	0.46
XOM				0.27	0.39	0.20	0.33	0.32	0.41	0.27	0.38
C					0.40	0.40	0.23	0.23	0.26	0.29	0.44
CB						0.28	0.37	0.33	0.38	0.34	0.43
FNMA							0.15	0.14	0.15	0.19	0.30
K								0.39	0.45	0.27	0.30
KO									0.40	0.24	0.29
PG										0.28	0.36
AET											0.30
GE											

- Correlation is estimated from the period, Nov 11, 2007 to Oct 28, 2009

Visualizing Correlation

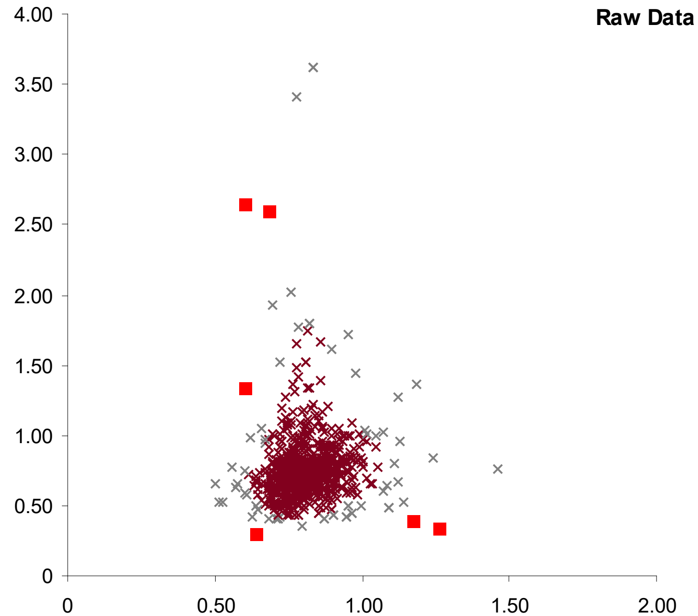


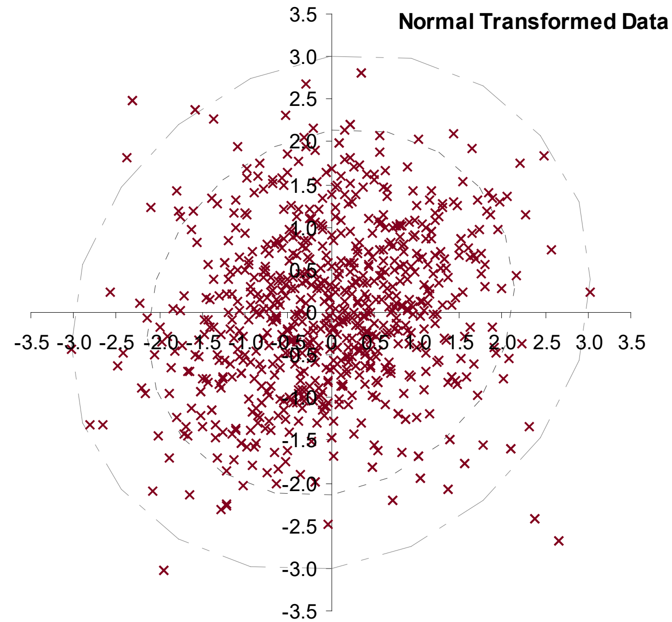
- Left: scatter plot of daily returns of AIG vs JPM with a 95% confidence ellipse
- Center: scatter plot of the same data transformed to standard normal
- Right: copula scatter plot of same data transformed to uniform (ranks)
- Normal transformed data gives clearest, most intuitive picture of association
  - Identifies and highlights uncorrelated but dependent, extreme tail correlation
- Daily returns are from Nov 11, 2007 to Oct 28, 2009

Detailed Analysis of Home to Pers. Auto Correlation

Some Indication of Heavy Tailed Dependence and "Pinching"

Private Passenger Auto Liability (x-axis) vs. Homeowners, $100M premium threshold **790 Annual Observations**





Association Summary

Linear Correlation, rho	1.7%
90% Confidence Interval	(-4.2%, 7.5%)
Base Linear Correlation	17.1%
Extreme Linear Correlation (n=72)	-15.4%
Rank Correlation	20.1%
Rank Correlation from rho	1.6%
Normal-Transformed Correlation	12.7%
Kendall Tau	11.5%
Rho from tau	18.0%
Outliers at 10% and 1% levels	9.1% and 1.4%

Univariate Summary

	Private Passenger Auto Liability	Homeowners
Mean	0.8133	0.7721
Min	0.5046	0.2957
Max	1.4601	3.6221
Std. Dev.	0.1005	0.3072
CV	12.4%	39.8%
Skewness	1.04	4.49
Kurtosis	3.89	31.72
90th percentile	93.4%	99.6%
99th percentile	112.9%	193.4%

Note: 1% outliers from normal copula marked in red. 10% and 1% and confidence intervals show on right.

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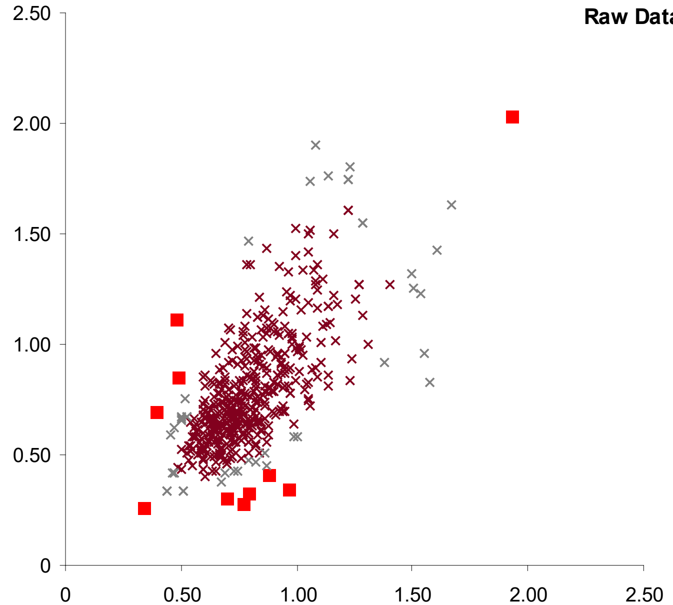
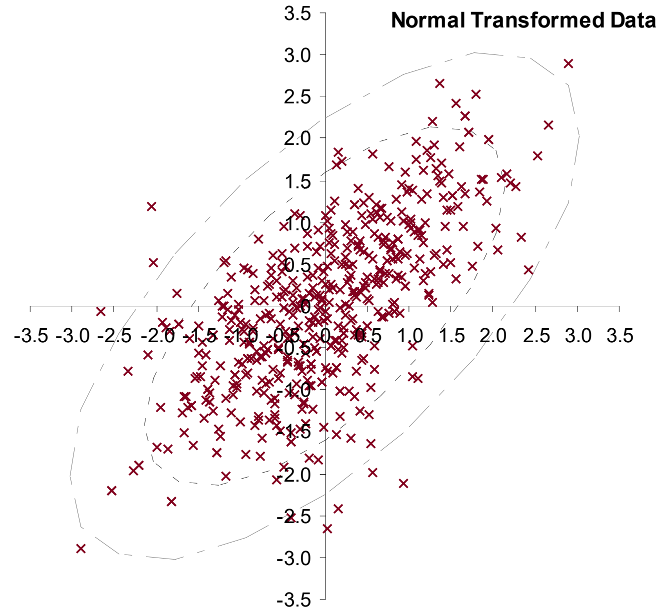
Commercial Auto Liability vs. Other Liability Occurrence

Commercial Auto Liability (x-axis) vs. Other Liability Occurrence, $100M premium threshold **514 Annual Observations**





Association Summary

Linear Correlation, rho	70.3%
90% Confidence Interval	(66.5%, 73.8%)
Base Linear Correlation	70.6%
Extreme Linear Correlation (n=52)	69.6%
Rank Correlation	70.4%
Rank Correlation from rho	68.6%
Normal-Transformed Correlation	66.9%
Kendall Tau	48.2%
Rho from tau	68.7%
Outliers at 10% and 1% levels	10.1% and 1.9%

Univariate Summary

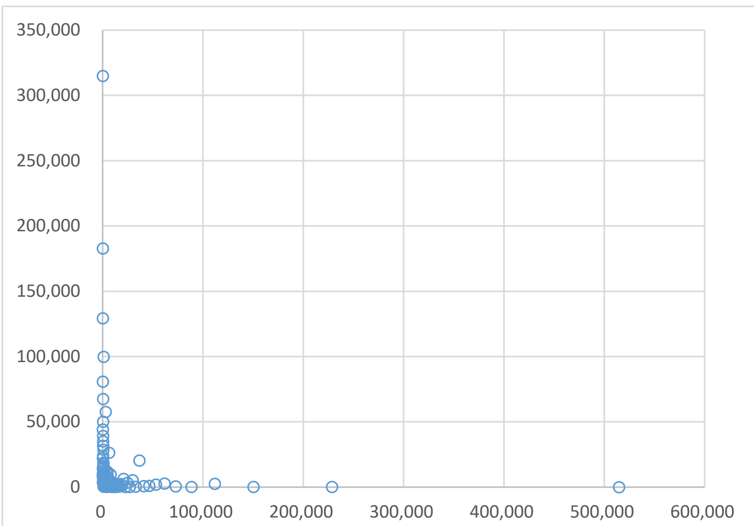
	Commercial Auto Liability	Other Liability Occurrenc
Mean	0.7999	0.7842
Min	0.3381	0.2659
Max	1.9288	2.0328
Std. Dev.	0.2053	0.2767
CV	25.7%	35.3%
Skewness	1.35	1.33
Kurtosis	3.37	2.30
90th percentile	105.5%	115.3%
99th percentile	153.5%	174.4%

Note: 1% outliers from normal copula marked in red. 10% and 1% and confidence intervals show on right.

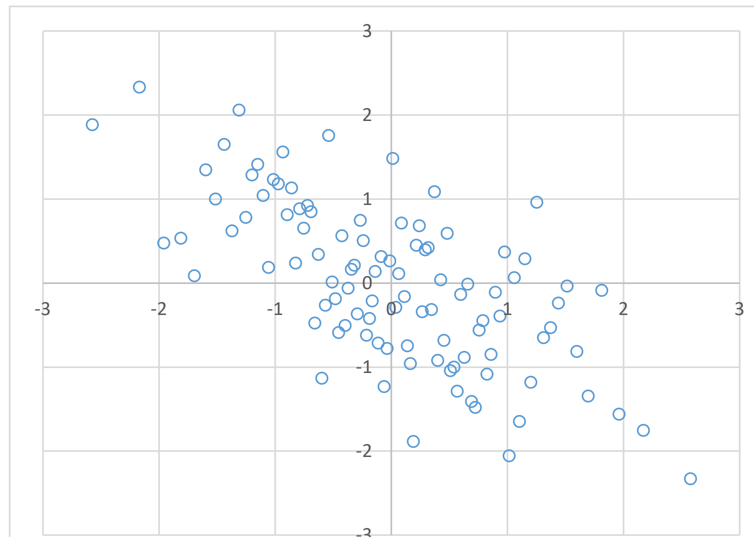
Correlation Diamond Reveals Relationships

	Corr(X,Y)	
Corr(X, N(X))		Corr(Y,N(Y))
	Corr(N(X), N(Y))	

	-11.1%	
55.1%		61.3%
	-67.5%	



Raw Data



Normal Transform

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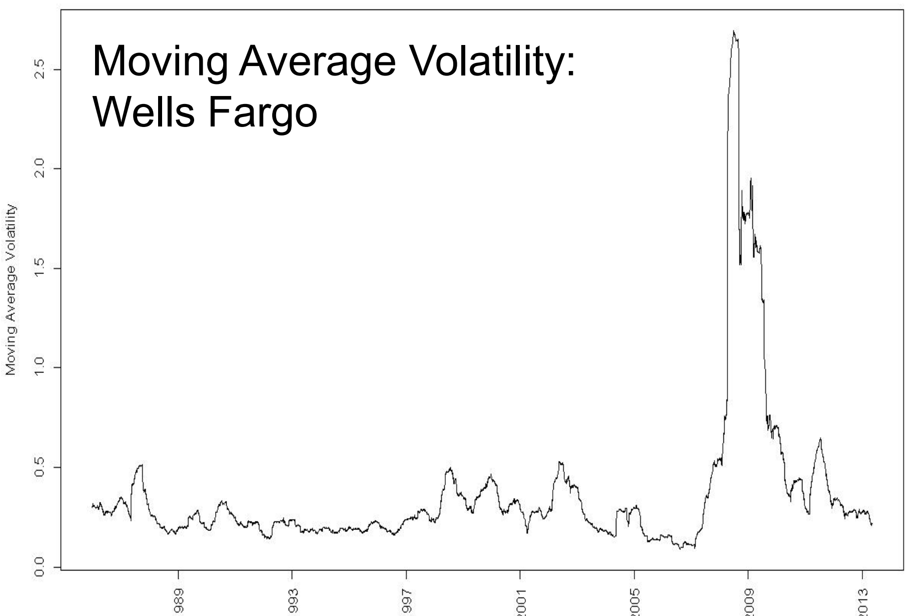
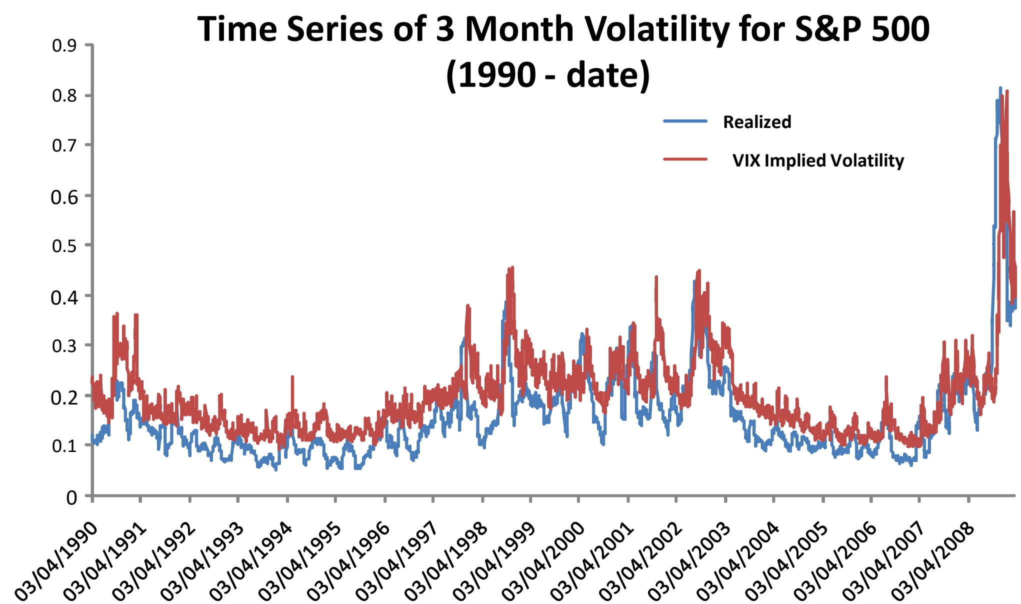
Stochastic Volatility and Stochastic Correlation

- Stochastic volatility models for stock market prices provide correction to geometric Brownian motion models well known to improve overall performance
  - Three month realized volatility on S&P 500 ranges from under 10% to over 80% during the last 20 years
  - Clear from historical data that volatility itself is an extremely volatile process





Stochastic Volatility Correction Meaningful

Equity Models During the Credit Crisis

- During September to March 2008, the S&P 500 declined by around 45%
- What probability do popular equity models attach to this type of market event ex-ante?

Annual Decline In Equities	Return Period for Model Calibrated in August 2008			
	GBM	Merton	Heston	Heston SVJD
45%	89,191	-	304	435
35%	1,441	1,261	79	106
30%	320	289	45	58
25%	94	85	26	32
20%	35	31	16	19
15%	16	14	10	12
10%	8	8	6	8

All models calibrated using maximum likelihood method

- Historical return perspective, between 1927 and 2008 (82 years)
  - Worst annual performance: -44.4% in 1931[1]
  - Second worst performance: -38.3% in 2008
  - Indicated return period between 1 in 40 and 1 in 80 years for 2008
  - Captured well by the Heston model

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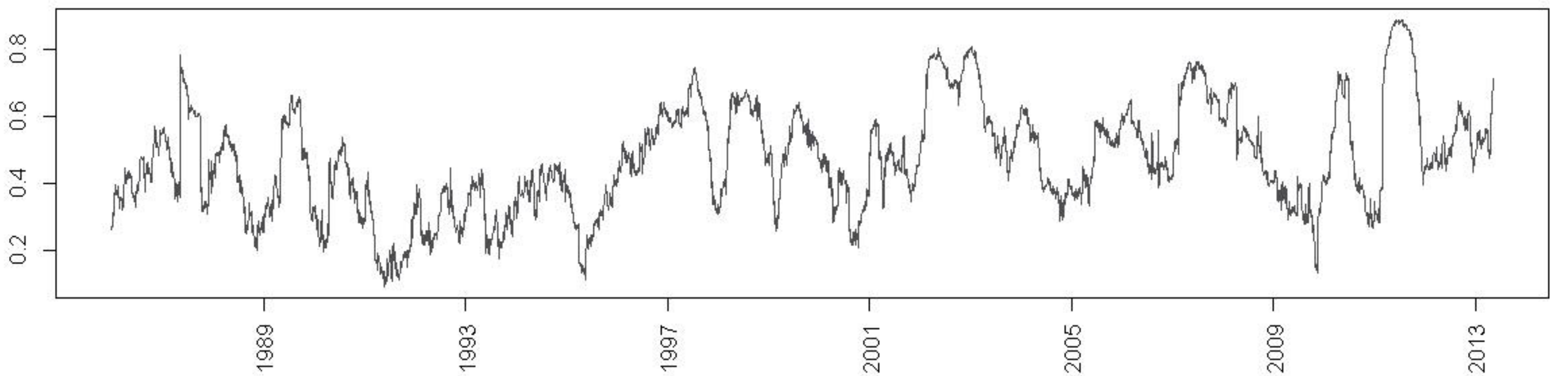
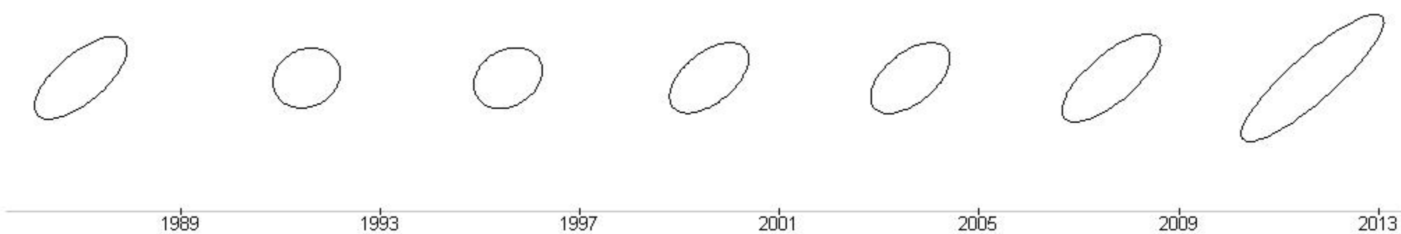
[1] Fama and French: How Unusual Was the Stock Market of 2008?

Correlation Also Volatile Over Time

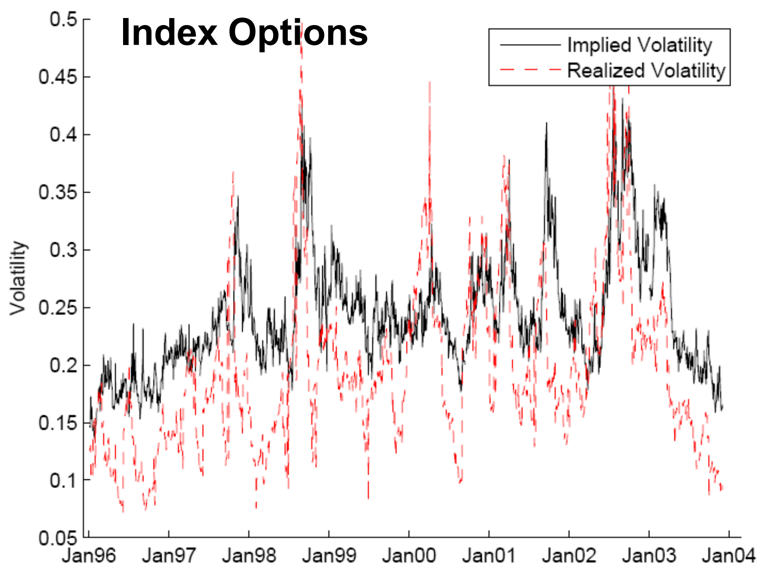


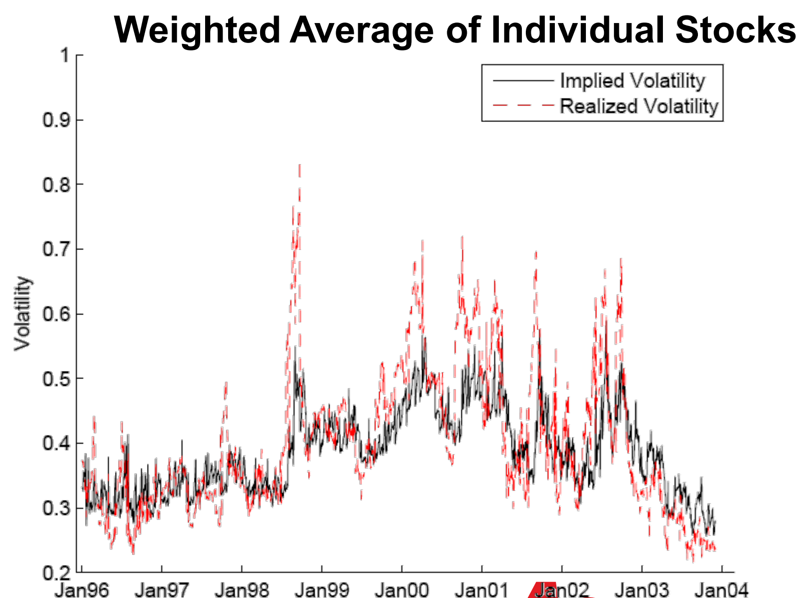


- Correlation of AIG and J.P. Morgan over time, with confidence ellipses below
- Higher correlation corresponds to lengthened ellipses
- Correlation time series is estimated using a 100 day moving window
- Ellipses are estimated from the first 100 days in Q3, every four years, starting at 1987

Correlation Risk: Individual Stock vs. Index Options

- Index options appear to “price volatility”: implied volatility > realized volatility
- Index volatility = sum of individual volatility + correlation adjustment
- Average realized volatility on individual stocks >~ average implied volatility
- Solution: correlation is a priced risk
  - Higher correlation decreases diversification benefits (bad)
  - Volatility not priced for individual stock options
- Cross-sectional studies of individual and index options confirm correlation is priced but individual stock volatility is not





Source: The Price of Correlation Risk: Evidence from Equity Options
by Driessen, Maenhout & Vilkov, Journal of Finance Volume: 64, Issue: 3, Published: June 2009

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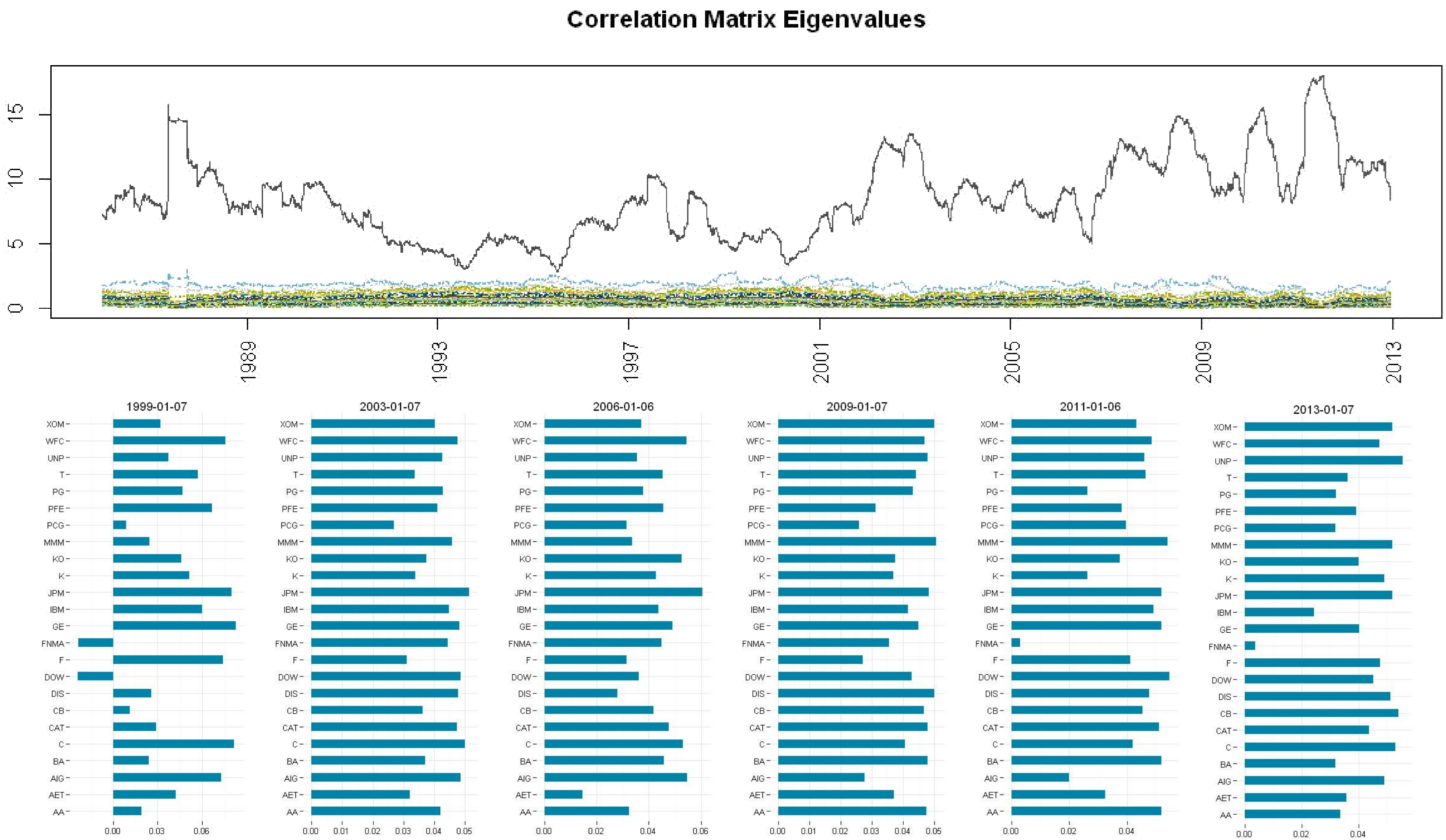
Eigen Evolution: Importance of Market



- Correlation matrices estimated with a 100 day moving window, starting at the given date
- Eigenvectors based on daily returns in a 100 business day window, starting at the first business day of the year

About Covariance Matrices

- A covariance matrix M is a symmetric positive definite
  - Meaning: $x^t M x > 0$ for all non-zero column vectors x
  - If V is a random vector with covariance matrix M then $x^t V = \sum_i x_i V_i$ has variance $x^t M x$
- Covariance matrices are "positive" and have "square roots" = Choleski decomposition = upper triangular matrix C so that $M = C^t C$
- If V is a row vector of mean zero independent variables then W = VC has covariance matrix M
  - $\mathrm{Cov}(W) = E[W^t W] = E[C^t V^t V C] = C^t E[V^t V] C = C^t C = M$
  - Positive definite iff covariance matrix
- A random symmetric matrix of high dimension is almost never positive definite, so how should we simulate them?

Facts about Covariance Matrices

- A covariance matrix M can be diagonalized
  - Exists orthogonal matrix B (origin preserving rigid motion, $B^t B = I$, $B^t = B^{-1}$) and a diagonal matrix D so that $M = B^t D B$
  - Diagonal elements of D are eigenvalues, matrix M is eigenvector basis
  - Positive definite iff all diagonal elements of D are positive: M is the basis expressing the quadratic form of M as a positive sum of squares

- Rugby ball ellipsoid shape determined by eigenvalues
  - Largest eigenvalue (longest axis) explains most variance in data
  - Next largest eigenvalue explains next most variance etc.
  - Eventually remaining axes almost circular (like a rugby ball vs. a pebble)…circular dimensions do not contribute to correlation or association in any way
  - The basis of Principle Components Analysis

- Correlation often driven by relatively few important parameters
  - Market risk for stocks is the classic example

How to Simulate Covariance Matrices

- Method 1: start with positive eigenvalues and rotate/reflect
  - Householder matrices provide random orthonormal matrices, reflection in plane perpendicular to random direction

- Method 2: compute the covariance matrix of a random sample from a multivariate distribution with assumed covariance structure
  - Distribution known for normal samples, Wishart distribution, with (reasonably) straight-forward density function
  - Iman Conover method can be used to generate sample
  - Generating full range of outcomes
  - Naturally centered around desired covariance matrix

- Method 3: Wishart process
  - Matrix analog of Cox-Ingersoll-Ross square root process
  - Continuous time evolution of family of covariance matrices

How to Simulate Covariance Matrices

- Method 4: simulate Choleski matrix C, and form $C^t C$
  - Upper triangular, otherwise free; hard to target around desired covariance matrix

- Method 5: averaging covariance/correlation matrices
  - Given covariance matrices A (base) and B (perturbation) form (A+B)/2 as the arithmetic mean or $B\,(B^{-1}A)^{1/2}$ as the geometric mean

- Method 6: exponential of a symmetric matrix, exp(M)

- Method 7: perturb a base matrix $M=C^tC$ with another correlation matrix P to form C^tPC
  - If P is close to independent then sampled matrices will be close to M

- Method 8: perturb base matrix M in a direction P via $M^{1/2}\exp(tM^{-1/2}PM^{-1/2})M^{1/2}$

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Method 9: Vines and Partial Correlations

- Consider three variables
  - G = crop growth rate
  - T = average temperature
  - R = rainfall

- Example, these are correlated as

	G	T	R
G	1	a > 0	b ? 0
T		1	c < 0
R			1

- Conditional correlation Corr(G, R | T) = $(gr - gt.rt) / ((1 - gt^2)(1 - rt^2))^{1/2}$

Method 9: Vines and Partial Correlations

- Vines and partial correlation provide convenient re-parameterization of correlation matrices because all parameters can be selected independently from (-1, 1)
- Using matrix below is particularly simple, called C-vine method
  - Generalization of the identity on previous slide can be used to “unpack” from conditional correlations to unconditional correlations
  - Unpack bottom to top, left to right

$$\begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\ \rho_{12} & 1 & \rho_{23;1} & \rho_{24;1} & \rho_{25;1} \\ \rho_{31} & \rho_{32} & 1 & \rho_{34;12} & \rho_{35;12} \\ \rho_{41} & \rho_{42} & \rho_{43} & 1 & \rho_{45;123} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & 1 \end{pmatrix}$$

H. Joe, Generating random correlation matrices based on partial correlations, Journal of Multivariate Analysis 97 (2006) 2177–2189

R and Matlab Code for Random Correlation Matrix

```r
RandUnifCorrMat <- function(n, eta)
{
## eta = 1 is uniform; otherwise proportional to
det^(eta-1)

## set up the partial correlations
  A <- array(0, dim=c(n,n))
  for(i in 1:n)  {
    A[i,i]=1
    beta = eta + (n-1)/2 - i/2
    if(i<n){
    for(j in (i+1):n)    {
      A[i,j] = 1 - 2*rbeta(1,beta,beta)
      A[j,i]=A[i,j]
       }
    } else
       A[n,n]=1
    }
## for each row, from bottom up; for each column, iterate
up the partial correlations to unconditional correlation

  for(i in (n-1):2){
    for(j in (i+1):n){
      for(k in 2:i){
        A[i, j] = A[i - k + 1, i] * A[i - k + 1, j] +
A[i, j] * sqrt((1 - A[i - k + 1, i] ^ 2) * (1 - A[i - k +
1, j] ^ 2))
      }}}

## make symmetric
  for(i in 1:(n-1))  {
    for(j in (i+1):n)    {
      A[j,i]=A[i,j] }}

  return(A)
}
```

```matlab
function A = RandUnifCorrMat(n, eta)
A = zeros(n,n);
for i = 1:n
    A(i,i)=1;
    beta = eta + (n-1)/2 - i/2;
    if(i<n)
        for j = (i+1):n
            A(i,j) = 1 - 2*betarnd(beta,beta,1,1);
            A(j,i)=A(i,j);
        end
    else
        A(n,n)=1;
    end
end

%% for each row, from bottom up; for each column, iterate
up the partial correlations to unconditional correl
for i = (n-1):-1:2
    for j = (i+1):n
        for k = 2:i
            A(i, j) = A(i - k + 1, i) * A(i - k + 1, j) +
A(i, j) * sqrt((1 - A(i - k + 1, i) ^ 2) * (1 - A(i - k +
1, j) ^ 2));
        end
    end
end
for i = 1:(n-1)
    for j = (i+1):n
        A(j,i)=A(i,j);
    end
end
```

Application of Vines Method

- Several alternative approaches
  - Starting with correlation matrix compute matrix of partial correlations and simulate partial correlations in a range around original matrix
  - Use one of Methods 2, 5, 7 or 8
- Example using Method 7
  - Perturb a base matrix $M=C^tC$ with another correlation matrix P to form C^tPC
  - Generate P using vines

Example Based on Non-Cat Insurance Risk

- Four standard commercial lines of business
  - Volatility and correlation assumptions from Insurance Risk Study, 2012 and 2013 editions

Correlation Volatility Example Assumptions

Line	Premium	Loss Ratio	E(Loss)	CV(Loss)	SD(Loss)
Comm Auto	200	70.0%	140	0.24	33.6
WC	500	75.0%	375	0.27	101.3
Other Liab Occ	400	65.0%	260	0.38	98.8
CMP	300	70.0%	210	0.36	75.6

Correlations	Comm Auto	WC	Other Liab Occ	CMP
Comm Auto	1	0.24	0.268	0.212
WC	0.24	1	0.244	0.172
Other Liab Occ	0.268	0.244	1	0.196
CMP	0.212	0.172	0.196	1

  - Portfolio standard deviation 204, CV 20.8%
  - Standard deviation if lines are independent is 164 (CV=16.6%) and if perfectly correlated 309 (CV=31.4%)
  - Average correlation factor is 0.22

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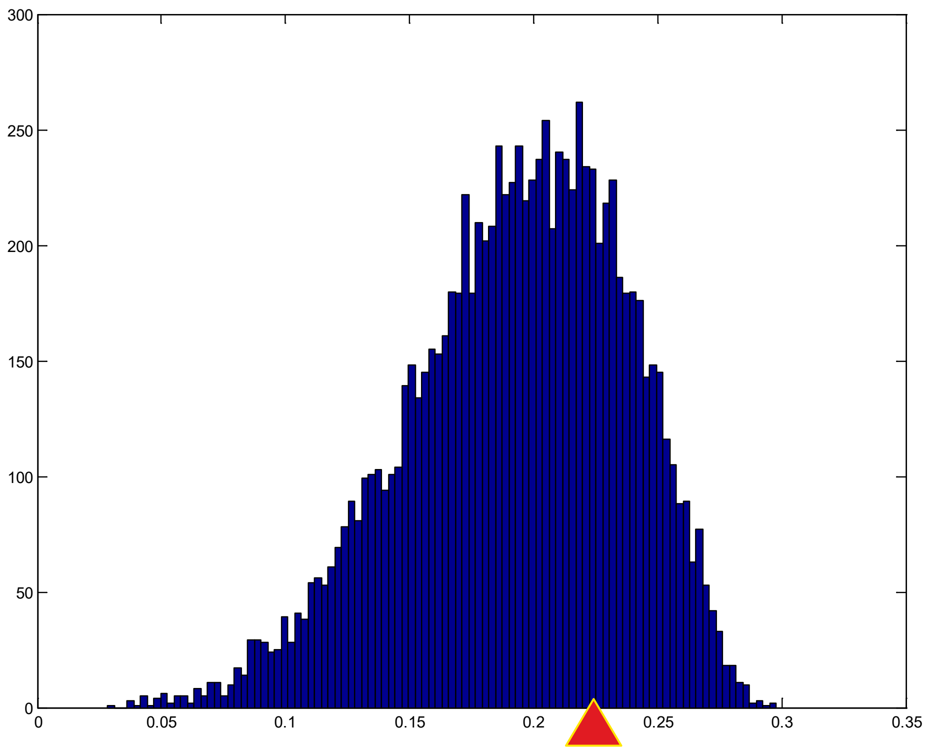
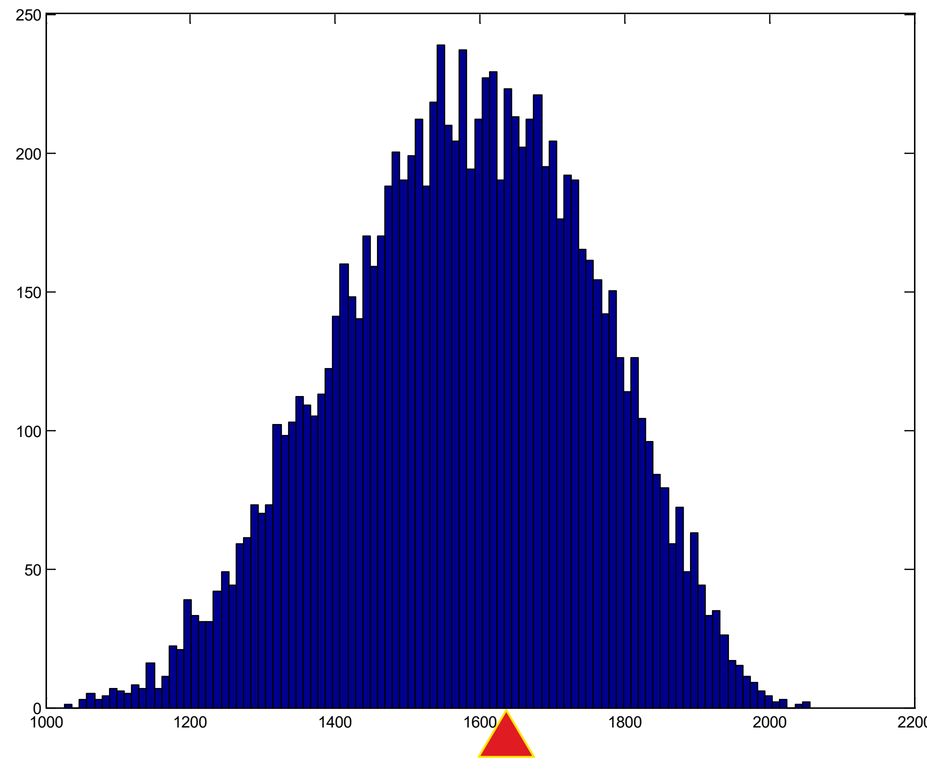
Example Based on Non-Cat Insurance Risk

Coefficient of Variation Distribution



Implied 200 Year PML



- 200 year PML on an expected basis is 1637, mid point of simulated range
- 90th percentile PML over the correlation distribution is 1803, 99th is 1931 and 99.5th is 1955

Conclusion

- Correlation is hard to estimate and non-constant
- Reflecting uncertainty and range of possibilities in correlation matrices improves tail estimates of models and provides a realistic stress test
- Once marginal distributions have been generated, applying Iman-Conover algorithm to variety of correlation matrices allows impact of uncertain correlation to be assessed quickly
- Method can be applied as a quick post-processing step to more rigorously modeled multivariate distributions
- Variety of methods to simulate correlation need further testing

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Aon Benfield
55 Bishopsgate
London EC2N 3BD
United Kingdom

tel: +44 (0) 20 7088 0044

fax: +44 (0) 20 7578 7001

www.aonbenfield.com